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General Introduction

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SOME GRAVITY WAVE PROBLEMS IN THE MOTION OF PERFECT LIQUIDS

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GENERAL INTRODUCTION

By W. G. PENNEY, F.R.S.

Five papers on various problems of gravity waves in perfect liquids are being published together; the first three papers are entirely theoretical, but the last two include a description of various experiments made to check and extend the mathematical investigations contained in the third paper.

Part I, by Penney & Price, considers the diffraction pattern produced by a semi-infinite straight breakwater inclined at any angle to the direction of approaching parallel harmonic infinitesimal sea waves. The spread of the waves into the lea of the breakwater is particularly interesting. The wave patterns behind gaps, and other extensions of the theory, are also developed.

Part II, by Penney & Price, attempts to deal with the extremely difficult mathematical problem of finite stationary periodic gravity waves in a perfect liquid of any depth. An alternative formulation of this problem can be made in terms of the periodic finite oscillations of a perfect liquid in a rectangular tank. No rigorous proof is obtained that periodic finite oscillations are possible, but at any rate, the equations can be solved by successive approximations, to any order required. Some comments are also made on the proof of Levi-Civita (1925) that irrotational progressive plane finite waves on deep water are possible. The proof as given only establishes the existence of small waves, none the less finite, and does not cover the more interesting cases of waves approaching in height those of the limiting form, treated first by Stokes (1880), where the crests are nodes of semi-angle

60°. The proof of the existence of periodic stationary finite waves appears to be much more difficult to develop than the proof of progressive finite waves. A discussion is also presented in part II of the existence of a limit to the wave height of stationary waves of permanent form.

Part III, by Penney & Thornhill, treats the motion of the collapse of a column of a perfect liquid resting on the bottom and surrounded by a second lighter liquid resting on the same bottom. The system is supposed to be released from rest. Various shapes of column, such as the hemi-cylinder or hemi-sphere, resting on the diametral plane, are considered, and sometimes, for simplicity, the lighter liquid is taken to be of zero density (i.e. no second liquid present). An approximate method, using characteristics, is found giving the motion for columns which are squat in form. A numerical solution, using 'relaxation' methods, for the collapse of a rectangular column is presented in the Appendix by Dr L. Fox and Dr E. G. Goodwin of the Mathematics Division of the National Physical Laboratory.

Part IV, by Martin & Moyce, describes experiments made to check and extend the mathematical results contained in part III. This paper confines itself to liquid columns released from rest in air, and the experimental results are compared with those given by the theory in the case where the second fluid is of zero, or at any rate negligible, density. The agreement between the experimental results and the theory is satisfactory.

Part V, by Martin & Moyce, uses a different technique from that described in the previous part. A method of releasing a liquid column from rest in a second liquid of lesser density is described, and the subsequent motion, as followed by ciné photography, is presented. A range of density ratios for the two columns was tried; and an attempt made to scale the results to the observations made on the 'base surge', appearing after the explosion underwater of the second atomic bomb at Bikini (see later photographs). It is concluded that no less than 120,000 tons of water were thrown into the air, mostly in the form of fine drops.

The first two papers were inspired by problems expected to be encountered in the use of the Mulberry harbours of Operation Overlord. The last three were suggested by the results of the second atomic bomb explosion at Bikini.

The theory of the behaviour of sea waves, when interrupted by breakwaters, acquired a special importance during the war in connexion with the construction of the Mulberry harbours. The obvious necessity for using the absolute minimum of material, time and labour in the construction of these temporary harbours made it imperative to have the most exact information possible, when solving the formidable engineering problems involved. In particular, the calculations relating to the breakwaters were required to be based on a theory made as exact as possible.

Some of the engineering problems arising in the conception of the Mulberry caissons are fully described in papers by Jellett, Pavry and Wood presented in 1947 at the Conference on War-time engineering problems, and published under the title *The civil engineer in war* (1948), sponsored by the Institution of Civil Engineers. Interesting new points were also brought out in the discussion. Another paper presented at the same conference by Lochner, Faber and Penney (1948) describes the Bombardon or floating breakwater. Parts I and II, now presented, made little or no contribution to the design of the breakwater structures, but they were of some assistance in the prognosis of wave conditions inside the harbours. There is no doubt that the 'infinitesimal theory' of the wave pressure, together with large safety factors, are sufficient for the practical approximations of civil engineering. In this sense, part II is too elaborate and academic for the practical engineer, but at least it is useful for estimating the errors in the 'infinitesimal theory' and thus revealing how much of the safety factors are to be taken up in covering purely mathematical uncertainties.

Parts III, IV and V relate to the 'base surge', one of the most terrifying features of an explosion of an atomic bomb in water. A full description of the base surge observed at Bikini, together with complete dimensions as functions of time, will be found in *The effects of atomic weapons*, a book sponsored by the United States Atomic Energy Commission (1950). A photograph of the base surge is printed in part V. Reproductions of other photographs of the base surge will be found in *Operation Crossroads* by the Historian of Joint Task Force One (1946).

A large underwater explosion throws water vertically upwards. The 'water column' is not solid water, but contains a considerable quantity of air. If the explosion is near to the surface, the water column contains relatively little water, and if the explosion is large, most of the water breaks up into drops due to the aerodynamic forces caused by the air. Soon after the explosion, the mixture of water drops and air collapses on to the sea, and provided the drops are fairly fine, the mixture of water drops and air moves almost as if it were a single fluid. Before long, of course, the water drops sediment downwards through the air on to the sea, but in the early stages, the differential falling motion of the water drops through the air can be neglected.

Any large explosion in shallow water would cause a base surge, but the reason why the phenomenon has singular importance for atomic explosions is that in this case the base surge contains all, or nearly all, of the deadly fission products.

The interrelationships between the five papers are as follows. Parts II and III are mathematically similar. Both deal with an irrotational motion of a perfect liquid and make series expansions for the velocity potential; both involve a moving unknown interface over which a boundary condition on pressure must be satisfied; both involve the time variable explicitly. Part I is related to part II in that both deal with breakwater problems, but the mathematical difficulties always encountered in diffraction problems precludes any attempt at dealing with finite waves in part I. Parts IV and V provide direct experimental evidence on the motions of the types considered in part III.

Finally, it is not out of place here to call attention to two similar mathematical points of great complexity in the following series of papers. The first point concerns the maximum amplitude of the finite stationary periodic wave in deep water; the second concerns the angle of contact at the ground of the expanding edge of a column of collapsing perfect incompressible fluid.

The maximum amplitude which the waves can have without losing their permanent periodic form is determined by a condition obtainable in several ways, explained fully in part III. In order to illustrate the mathematical difficulties encountered in the problem, however, we confine attention here only to one form of the limiting condition, namely, that the amplitude of the waves must be such that the wave motion never requires the maximum downward acceleration at any point in the free surface to exceed g; otherwise this part of the surface is unstable.

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Instability first appears in the motion for arbitrary wave height at the crest at the instant of greatest height. Now the elementary theory of infinitesimal stationary waves leads to the following formula for the surface elevation:

$$Y = (A\lambda/2\pi)\sin\{(2\pi g/\lambda)^{\frac{1}{2}}t\}\cos 2\pi x/\lambda. \tag{1}$$

The criterion of stability is considered to be

$$-\ddot{Y} \leqslant g. \tag{2}$$

Hence from (1), at the crest ($x = 0$) at its greatest height, we find that

$$A \leqslant 1. \tag{3}$$

Admittedly, this value of A is far from infinitesimal. However, another approximate limiting value of A can be found from the velocity potential.

The velocity potential used in obtaining (1) is

$$\phi = g^{\frac{1}{2}}(\lambda/2\pi)^{\frac{3}{2}} A\, e^{2\pi y/\lambda}\cos\{(2\pi g/\lambda)^{\frac{1}{2}}t\}\cos 2\pi x/\lambda. \tag{4}$$

The upward vertical velocity v is $-\partial\phi/\partial y$. The acceleration at the crest at its greatest height is easily shown from (4) to be $-gA\,e^{A}$. For stability, we require this to be greater than $-g$. Hence, for stability,

$$A\,e^{A} \leqslant 1. \tag{5}$$

The maximum value of A by this approach is 0·567.

A simple physical argument which destroys confidence in either of the results (3) or (5) can be advanced. According to the infinitesimal theory of stationary waves, and, indeed, according to our theory valid to any order, the periodic stationary wave motion can be generated from rest by imagining the surface modulated to the correct shape and the constraining surface suddenly removed. Now, the condition of stability so far postulated is that the downward acceleration must not exceed g. The question arises as to how it is possible for any particle in the surface of a mass of fluid with a modulated surface at zero pressure, released from rest, to acquire a downward acceleration greater than or equal to g. We conclude that the crest of the maximum wave at its greatest height is a singular point and that it is a node enclosing a total angle of 90°. The slope at the crest is therefore $\pm 45°$, and the initial downward acceleration at the node is just g.

We have now run into a formidable mathematical problem. The first-order theory (i.e. the infinitesimal theory) gives a smooth cosine profile; any higher order theory to finite order also gives a horizontal gradient at the peaks. To find the limiting wave height we require the gradient of the profile to be discontinuous at the peaks. The mathematical difficulty is to some extent avoided by obtaining from the hydrodynamical equations to any order the downward acceleration at the peaks. By equating this to g, we obtain a reasonably accurate expression for the wave parameter A. Our equations to the fifth order give us the profile accurately everywhere except near the peak, give the height of the peak with fair accuracy, but give the slope at the peak zero instead of 45°. The value of A lies between the values given in (3) and (5) above, but the wave profile at the tip is distinctly different in shape.

The second difficult mathematical point appears in part III. The angle of contact at the ground of the collapsing column can be made initially any value between 0 and π. Professor

H. Jeffreys has kindly supplied a proof that in the steady state, the angle of contact must be either 60° or 0; and experiments on 'gravity currents' mentioned by von Karman (1940) suggest that 60° is the true angle. In our work, we assume that the shape of the surface of the collapsing column can be expanded either in terms of orthogonal functions (cosines or Legendre polynomials) or as a power series in the polar angle θ. With the cosine or Legendre expansions, the angle of contact remains 90° with any finite expansion. With four or five term expansions in θ, however, the angle of contact does not seem to be approaching 60°. Indeed, the angle of contact is getting continuously smaller, and is apparently approaching zero. The numerical solution of Fox & Goodwin also suggests a limiting angle of zero. Possibly, the limiting angle really is zero for a column collapsing *in vacuo* but is 60° for a column collapsing in a second fluid. Another possible explanation is that the expansions must include many more terms than we have been able to manage before such a fine detail as the angle of contact at the ground is adequately represented.

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